COMBINED ANALYSIS OF INCOMPLETE BLOCK DESIGNS

By S. S. Swaminathan

The Rubber Board, Kottayam, Kerala

AND

M. N. Das

Indian Council of Agricultural Research, New Delhi

Introduction

In agriculture as well as in industry, it frequently happens that the experimenters have to lay out their experiments at different places or at different plants, possibly with some treatments common. It is also possible that the research workers may conduct the experiments in the same place but at different times with one or more treatments common to the whole set. In all these cases, the precision of estimates of the treatment effects can be improved by having a combined analysis of these experiments. Gomes and Guimaraes (1958) have considered the case when the individual experiments are laid out in randomised complete block designs. Pavate (1961) has considered the case when the individual experiments are laid out in B.I.B. designs each with the same parameters. In the analysis presented by him, the error sum of squares contains the component of between treatment × place interaction. In this paper a method of intrablock analysis in the case when the parameters of the B.I.B. designs are different but one or more treatments are common in all these experiments has been given.

Let there be g B.I.B. experiments, *viz.*, $E_1, E_2 \ldots\ldots E_g$, to be analysed jointly. With usual notations, let the parameters of E_i be denoted by $(v_i, b_i, r_i, k_i, \lambda_i)$. Further let us assume that there are c treatments common in all these experiments. Hence there are $v = \sum_i (v_i - c) + c$ different treatments in all. Following Gomes and Guimaraes, let us call the c treatments (common to all the experiments) 'common' treatments, and the remaining $\sum_i (v_i - c)$ treatments, 'regular' treatments. Denoting the common treatments by $t_1, t_2, \ldots t_c$ and the regular treatments in the i-th experiment by $t_1^i, t_2^i, \ldots t^i_{v_i-c}$ we have the analysis as follows;

On the model

$$y_{ij} = \mu + t_i + b_j + e_{ij}$$

where t_i is the effect of the i-th treatment, b_j is the effect of the j-th block, μ is the general component and e_{ij} is a random component with expectation Zero and Variance σ^2, the e_{ij}'s being distributed independently, the normal equations for estimating the treatment effect after eliminating the block constants come out as below:

$$\left.\begin{aligned} &\left(\sum_i r_i - \sum \frac{r_i}{k_i} + \sum \frac{\lambda_i}{k_i}\right) t_s - \sum_i \frac{\lambda_i}{k_i}\left(\sum^{c} t_l\right) \\ &\qquad - \sum_i \frac{\lambda_i}{k_i} \sum_j^{v_i - c} t_j{}^i = Q_s \\ &s = 1,\ 2,\ \ldots c \\ \text{and} \\ &\left(r_i - \frac{r_i}{k_i} + \frac{\lambda_i}{k_i}\right) t_j{}^i - \frac{\lambda_i}{k_i}\left(\sum^{c} t_l + \sum_j^{v_i - c} t_j{}^i\right) = Q_j{}^i \\ &i = 1,\ 2,\ \ldots g,\ j = 1,\ 2,\ \ldots v_i - c, \end{aligned}\right\} \quad (1)$$

where Q's are the adjusted treatment totals, *e.g.*, Q_s is the adjusted treatment total for the s-th treatment which is defined as follows: Let T_s be the total for the s-th treatment, *i.e.*, T_s is the sum of the observations from the $\sum_i r_i$ experimental units to which s-th treatment has been applied, and $B_j{}^i$ is the total of the j-th block in the i-th experiment, *i.e.*, $B_j{}^i$ is the sum of the observations from the k_i experimental units in the j-th block in the i-th experiment. Then the adjusted treatment totals Q_s for the s-th common treatment is given by

$$Q_s = T_s - \sum_{i=1}^{g} \frac{1}{k_i} \sum_j B^i{}_{j(s)} \qquad (2)$$

and for a regular treatment the adjusted treatment total is given by

$$Q_j{}^i = T_j{}^i - \frac{1}{k_i} \sum_l B^i{}_{l(j)} \qquad (3)$$

where $T_j{}^i$ is the sum of the observations from the r_i experimental units to which j-th treatment has been applied in the experiment, The normal equations thus obtained are not all independent and to render the solution of the normal equations unique we impose a further restriction, *viz.*,

$$\left(\sum^{g}\frac{\lambda_i}{k_i}\right)\left(\sum^{c} t_s\right) + \sum^{g}\frac{\lambda_i}{k_i}\sum_{j}^{v_i-c} t_j^{\,i} = 0 \tag{4}$$

using this condition the first s equations of (1), *i.e.*, those for the common treatments can be written as

$$\left(\sum r_i - \sum \frac{r_i}{k_i} + \sum \frac{\lambda_i}{k_i}\right) t_s = Q_s. \tag{5}$$

$s = 1, 2, \ldots c$

Denoting

$$r_i - \frac{r_i}{k_i} + \frac{\lambda_i}{k_i} \quad \text{by} \quad r_i E_i \quad \text{we have}$$

$$\left(\sum_i r_i E_i\right) t_s = Q_s \quad s = 1, 2, \ldots c. \tag{6}$$

The normal equation corresponding to j-th regular treatment in the i-th experiment is given by

$$r_i E_i t_j^{\,i} - \frac{\lambda_i}{k_i}\left(\sum^{c} t_s + \sum_{j}^{v_i-c} t_j^{\,i}\right) = Q_j^{\,i} \tag{7}$$

Summing this equation over $j = 1, 2, \ldots v_i - c$ we have

$$r_i E_i \sum_j t_j^{\,i} - \frac{\lambda_i}{k_i}(v_i - c)\sum^{c} t_s - \frac{\lambda_i}{k_i}(v_i - c)\sum t_j^{\,i} = \sum_j Q_j^{\,i}. \tag{8}$$

Therefore

$$\sum_j t_j^{\,i} = \frac{\sum Q_j^{\,i} + \frac{\lambda_i}{k_i}(v_i - c)\sum^{s} t_s}{r_i E_i - \frac{\lambda_i}{k_i}(v_i - c)}. \tag{9}$$

Hence on further simplifications

$$r_i E_i t_k^{\,i} = Q_k^{\,i} + \frac{r_i E_i}{c\sum_i (r_i E_i)}\sum^{s} Q_s + \frac{\sum_j Q_j^{\,i}}{c} \tag{10}$$

$j = 1, 2, \ldots v_i - c, \; i = 1, 2 \ldots g$

Adjusted treatment $S.S. = \sum^{c} \hat{t}_s Q_s + \sum^{g} \sum^{v_i - c} t_j^i Q_j^i$

The variances of various treatment differences are given below

1. For treatments belonging to the common group

$$V(\hat{t}_i - \hat{t}_j) = \frac{2\sigma^2}{\sum_i r_i E_i}.$$

2. For treatments one of which is regular and belongs to experiment

$$V(\hat{t}_i - \hat{t}_j^i) = \left[\frac{(c-1)}{c \sum r_i E_i} + \frac{c+1}{c r_i E_i}\right]\sigma^2.$$

3. For treatments both of which are regular from i-th and j-th experiment

$$V(\hat{t}_j^i - \hat{t}_j^{i'}) = \frac{2(c+1)}{c r_i E_i}\sigma^2.$$

Now the treatment (common) × Places interaction can be found out as usual in orthogonal data from the common treatment × places table. This has got $(c-1)(g-1)$ degrees of freedom.

The final analysis of Variance table can be put as follows:

Source	d-f	S.S.
Blocks	.. $\Sigma b_i - 1$	$\sum\sum \frac{B_j^{i2}}{k_i} - C.F.$
Treatments	.. $\Sigma(v_i - c) + c - 1$	$\Sigma t_j^i Q_j^i$
Treatment (common) × Places	$(c-1)(g-1)$	As usual
Error	.. By subtraction	By subtraction
Total	.. $\Sigma v_i r_i - 1$	$\Sigma y_{ijk}^2 - C.F.$

From this table the significance of treatment effects can be tested as usual. In this case, since the interaction $S.S.$ is separated from the error SS, an inference regarding treatment × Places interaction will be available.

Special Cases

(i) When $v_i = v$, $r_i = r$, $k_i = k$, $b_i = b$ and $\lambda_i = \lambda$, $i = 1, 2, \ldots g$ the above case reduces to that of the design considered by Pavate (1961)..

(ii) If $k = v$ in the above case the B.I.B. reduces to that of the randomised block design and hence the analysis reduces to the case considered by Gomes and Guimaraes (1958).

(iii) When $v_i = v$, $r_i = r$, $k_i = k$, $b_i = b$, $\lambda_i = \lambda$ and $c = v$, $i = 1, 2, \ldots g$ this reduces to the case of a B.I.B. experiment repeated g times. The analysis reduces to the groups of experiment of B.I.B. designs.

Group Divisible Designs

In the last section we considered the combined analysis of group of experiments in Balanced Incomplete Block designs, the designs being all different but some treatments are common in all the experiments. As a first step towards the extension of this result to Partially Balanced Incomplete Block designs, a subclass of P.B.I.B. designs, *viz.*, G.D. designs has been considered in this section. The problem is given below.

Let there be g, G.D. experiments, *viz.*, $E_1, E_2, \ldots E_g$ to be analysed jointly. Let the following be the parameters:

$v_i = v = nm$ .. Number of treatments (n groups each containing m treatments).

$b_i = b$ .. Number of blocks.

$r_i = r$ .. Number of replications for each treatment.

$k_i = k$ .. Number of units per block.

$\lambda_1^i = \lambda_1$ .. Number of times any two treatments belonging to the same group occur together in a block in the design.

$\lambda_2^i = \lambda_2$ .. Number of times any two treatments belonging to different groups occur together in a block in the design.

Let us suppose that one group of treatments is common in all these experiments. Hence there are $(n - 1)\,mg + m$ different treatments in all. Let us call the common treatments as $t_{11}, t_{12}, \ldots t_{1m}$ and the regular treatments in the i-th experiment and j-th group as t_{jk}^i, $k = 1, 2, \ldots m$, $j = 2, 3, \ldots n$, $i = 1, 2, \ldots g$.

Let y_{ijk} be the observed result applying ij-th treatment to the k-th block. We may write with usual notations,

$$y_{ijk} = \mu + t_{ij} + b_k + e_{ijk}$$

so that if the residuals e_{ijk} are homoscedastic then the intra-block estimate of any linear treatment contrast is obtained by substituting in the contrast the solution of normal equations,

$$\left(gr - \frac{gr}{k} - \frac{g\lambda_1}{k}\right) t_{1j} - \frac{\lambda_1 g}{k} \sum_j t_{1j} - \frac{\lambda_2}{k}\left[\sum_i^g \sum_p^n \sum_q t_{pq}^i\right] = Q_{ij} \tag{11}$$

for common treatments, and

$$\left(r - \frac{r}{k} + \frac{\lambda_1}{k}\right) t_{pq}^i - \frac{\lambda_i}{k} \sum_q t_{pq}^i$$

$$- \frac{\lambda_2}{k}\left[\sum_j^n \sum_k t_{jk}^i + \sum_j t_{1j}\right] = Q_{pq}^i \tag{12}$$

for regular treatments, where Q's denotes the adjusted treatment totals. Denoting

$$r - \frac{r}{k} + \frac{\lambda_1}{k} \text{ by } rE \text{ and } \sum_j t_{ij}^k \text{ by } G_i^k$$

the above equations corresponding to common treatment reduces to

$$gr\,Et_{ij} + \frac{g(\lambda_2 - \lambda_1)}{k} G_1 - \frac{\lambda_2}{k}\left[gG_1 + \sum_i^g \sum_k^n G_k^i\right] = Q_{ij} \tag{13}$$

$j = 1, 2, \ldots m$

putting the restriction

$$gG_1 + \sum_i \sum_j G_j^i = 0 \tag{14}$$

the solution of t_{ij} reduces to

$$t_{ij} = \frac{1}{grE}\left[Q_{1j} + \frac{(\lambda_1 - \lambda_2)}{nm\lambda_2} \sum_s Q_s\right] \tag{15}$$

$= 1, 2, \ldots m$

For regular treatments, equation (12) reduces to

$$rEt_{pq}^i = \frac{\lambda_1}{k} G_p^i - \frac{\lambda_2}{k}\left(G_1 + \sum_{j \neq p} G_j^i\right) = Q_{pp}^i \tag{16}$$

$q = 1, 2, \ldots m;\ p = 2, 3, \ldots n;\ i = 1, 2, \ldots, g.$

Summing Equation (16) over q we have

$$rEG_p{}^i - \frac{n\lambda_1}{k} G_p{}^i - \frac{n\lambda_2}{k} \sum_{j \neq p} G_j{}^i = \sum_q Q_{pq}{}^i + \frac{n\lambda_2}{k} G_1 = P_p{}^i. \quad (17)$$

$p = 2, \ldots, n; \quad i = 1, 2, \ldots g$

Summing Equation (17) over $p = 2, 3, \ldots n$, we have

$$\sum_{i=2}^{m} G_j{}^i = \frac{\sum_{j=2}^{m} p_j{}^i}{\left[rE - \frac{n\lambda_1}{k} - \frac{(m-2)\, n\lambda_2}{k} \right]} = R^i \quad (18)$$

On solving Equation (2) corresponding to regular treatment we have

$$t_{pq}{}^i = \frac{1}{rE} \left[Q_{pq}{}^i + \frac{(\lambda_1 - \lambda_2)}{nm\lambda_2} \sum_j Q_{pj}{}^i + \frac{krE}{n^2 gm\lambda_2} \sum_j Q_{1j} + \frac{krE}{n^2 m\lambda_2} \sum_{p=2}^{m} \sum_{j=1}^{n} Q_{pj}{}^j \right] \quad (19)$$

$q = 1, 2, \ldots, m; \quad p = 2, 3, \ldots, n; \quad i = 1, 2, \ldots, g.$

The variances of various treatment differences are given below:

1. Variance of the difference of two treatments both of which are common:

$$V(\hat{t}_{1j} - \hat{t}_{1j'}) = \frac{2\sigma^2}{grE}.$$

2. Variance of difference of two treatments one of which is common

$$V(\hat{t}_{1j} - \hat{t}_{pq}{}^k) = \left[\frac{(g+1)}{grE} \left(1 + \frac{\lambda_1 - \lambda_2}{nm\lambda_2} \right) + \frac{k(g-1)}{gn^2 m\lambda_2} \right] \sigma^2$$

3. Variance of the difference of two treatments both of which are regular and belong to the same group:

$$V(\hat{t}_{pq}{}^i - \hat{t}_{qq'}{}^j) = \frac{2\sigma^2}{rE}.$$

4. Variance of the difference of two treatments both of which are regular but belonging to different groups in the same experiment:

$$V_{p \neq p'}(\hat{t}_{pq}{}^i - \hat{t}_{p'q'}{}^i) = 2 \left[\frac{1}{rE} + \frac{\lambda_1 - \lambda_2}{rEnm\lambda_2} \right] \sigma^2.$$

5. Variance of the difference of two treatments both of which are regular and belong to different experiments

$$V(\hat{t}_{pq}^{\,i} - \hat{t}_{p'q'}^{\,i'}) = 2\left(\frac{1}{rE} + \frac{\lambda_1 - \lambda_2}{rEnm\lambda_2}\right)\sigma^2.$$

The above results can be extended in a similar way when two or more groups are common between these experiments.

As a special case if there are c groups common in all these experiments and if the group size $m = 1$ then this reduces to the case considered by Pavate (1961) and the corresponding results can be deduced from this.

SUMMARY

A combined analysis of different balanced incomplete block designs with common treatments has been given in this paper. This case has been extended to the combined analysis of a number of group divisible designs (with same parameters) when one or more groups of treatments are common in all these designs.

ACKNOWLEDGEMENTS

We are highly thankful to Dr. G. R. Seth and Dr. V. G. Panse, Statistical Adviser, Indian Council of Agricultural Research, for providing us with adequate facilities for our investigation.

REFERENCES

1. Gomes, F. P. and Guimaraes, R. F. .. "Joint analysis of experiments in complete randomised block with some common treatments," *Biometrics*, 1958, **11**, 521–26.
2. Pavate, M. V. .. "Combined analysis of B.I.B. designs with some common treatments," *Ibid.*, 1961, **17**, 111–19.

9